liminary notice, and defer to a subsequent occasion a detailed account of the whole subject, as known from observation. In that paper I purpose to show how the indices can be accurately determined in minerals of different structure, to point out how closely observation agrees with theory, and to describe a number of anomalies met with in particular specimens.

X. "On the Foci of Lines seen through a Crystalline Plate." By G. G. Stokes, M.A., Sec. R.S. Received June 21, 1877.

At the Soirée of the Royal Society on the 25th of April Mr. Sorby showed me the method he had recently devised for discriminating between minerals by focusing a microscope over a delicate image of cross lines, which image was viewed, first directly, and then through a crystalline plate, having previously been adjusted to be at the distance of the lower surface of the plate. With glass and singly refracting substances the alteration of the focus produced by the interposition of the plate affords a measure of its refractive index. But with a plate cut from a doubly refracting crystal, not only is there more than one focal distance, but for one at least of the pencils there is (except in special cases) no true focus, but the foci of the two systems of cross lines are found at two different depths, or else there is no sharply defined image at all, according to the orientation of the lines relatively to lines fixed in the crystalline plate. Moreover the result obtained on applying the formula which, for a singly refracting plate, gives the refractive index from the measured displacement of the focus is often widely different from what is known to be the refractive index of the crystal, for the pencil under examination, in a direction perpendicular to the plate.

The phenomena will be described in detail by Mr. Sorby in his own paper. My object is to show how they flow from the known laws of double refraction, as consequences of which they will necessarily come under review*.

---

* [It seemed pretty certain that some of the phenomena must have been noticed before, though I am not aware that they have been described, or their theory worked out in any detail. I find that Prof. Clifton has been in the habit of using an instrument somewhat similar to Mr. Sorby's, which was procured several years ago for the Museum of the University of Oxford, and that he was familiar with such things as the low apparent index of calcite for the extraordinary pencil nearly in the direction of the axis, and the astigmatism in general of a pencil refracted across a plate otherwise than by ordinary refraction; and, further, that he utilized these phenomena for the instruction of students as to the general form of the wave-surface. No one, however, so far as I know, before Mr. Sorby, had applied the phenomena to the practical discrimination of minerals, or had worked them out quantitatively and in detail; and it is my desire to complete the subject, by supplying the mathematical theory, that must be my excuse for offering to the Society an investigation which in itself consists merely in easy deductions from well-known principles.

It is perhaps hardly necessary to refer to a paper by Dr. Quincke in Poggendorff's

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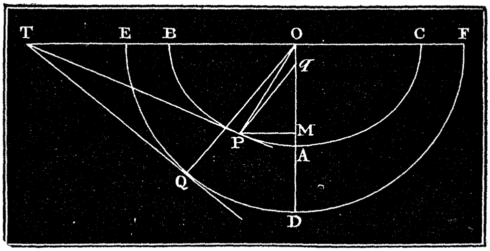
The simplest case is that of a uniaxal crystal, such as Iceland spar, cut perpendicular to its axis. As regards the ordinary ray, a plate cut from a uniaxal crystal, in whatever direction, behaves, of course, like a plate of glass, so far as focusing is concerned, and the index obtained is the true ordinary index. To find what takes place as regards the extraordinary ray, we must have recourse to Huyghens's construction.

Let O be any point in the further surface of the crystalline plate, OA perpendicular to the surface the direction of the axis, OP the direction of any extraordinary ray. Let the plane of the paper be the plane of incidence, AOP; take OA to represent the velocity of propagation (a) within the crystal in the direction of the axis, and OD in OA produced to represent the velocity of propagation (unity) in air. With O as centre, construct the half-spheroid, BAC, which is the extraordinary sheet of the wave-surface, and the hemisphere EDF representing the wave into which a disturbance emanating from O would have spread in air in a unit of time, and let OB or OC be denoted by c. Let OP cut the half-spheroid in P. At P draw a tangent plane to the spheroid, the trace of which on the surface of the crystal is projected in T; and through the trace T draw a tangent plane to the hemisphere, touching it in Q, and join OQ. Then if an extraordinary ray travel within the crystal in the direction OP, the refracted ray to which it will give rise will travel in a direction parallel to OQ. Hence if we now take OP to denote the whole path of the ray within the plate, and draw Pq parallel to QO, cutting OA in q, the ray OP, after refraction at P, will proceed as if it came from q. Hence the limiting position of q, as P moves up to A, will be the geometrical focus, after refraction, of a small pencil proceeding from O, and having OA for its axis.

Fig. 1.



Draw PM perpendicular to OA, and let m represent the ratio of the sine of refraction to the sine of incidence. Then

$$m = \mathrm{OP} : \mathrm{P}q\,;$$

'Annalen' (1862, vol. xxvii. p. 563), containing some elaborate observations on the focal lines formed within a doubly refracting plate, as his experiments were made in a very different manner from those of Mr. Sorby, and with a totally different object in view.—*October* 1877.]

and by similar triangles

$$\mathrm{P}q : \mathrm{PM} = \mathrm{OT} : \mathrm{OQ} = \mathrm{OB}^2 : \mathrm{PM} \, . \, \mathrm{OQ},$$

since

$$\mathrm{OB}^2 = \mathrm{OT} \, . \, \mathrm{PM};$$

also

$$\mathrm{OP} = \mathrm{OA}$$

ultimately. Hence as

$$\mathrm{OA} : \mathrm{OB} : \mathrm{OQ} = a : c : 1,$$

we have ultimately

$$m = \frac{a}{c^2} = \frac{\mu'^2}{\mu}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ, μ' denote the ordinary and the principal extraordinary indices of refraction, which are the reciprocals of a, c.

In this particular case the ordinary and extraordinary images cannot be distinguished directly by their polarization, since each consists of rays polarized in all azimuths. But if the objective of the microscope be limited by a narrow aperture, so as to give a predominance to rays lying in one plane, there will in the ordinary image be a predominance of polarization in a plane parallel to the length of the aperture, and in the extraordinary image of polarization in the perpendicular plane.

Next take the case of a uniaxal crystal cut parallel to the axis. In this case, as regards the extraordinary pencil, the divergence after refraction will be different in the axial and equatorial planes, so that a small pencil diverging from a point at the under surface of the crystal will, after refraction, diverge from two focal lines; and in order that a line may be seen distinctly, it must lie in one of the planes of symmetry, in which case, at a certain focal adjustment of the microscope, each element of the line would be seen as a short line in the direction of the actual line, and therefore the line as a whole will be seen sharply defined.

In the equatorial plane the extraordinary ray obeys the ordinary law of refraction; and as regards divergence, therefore, in this plane, on which depends clear vision of a line parallel to the axis, the apparent index will be the same as the real index, μ'. In the axial plane every thing will be the same in respect of divergence as in the first case, except that the principal axes of the ellipse which is the section of the extraordinary wave-surface will be interchanged. Accordingly a line in the equatorial plane will be seen distinctly at a focal adjustment which will give an apparent refractive index $\mu^2 : \mu'$.

There will therefore, on the whole, be three focal adjustments of the microscope at which one or other of the systems of cross lines, or both together, will be seen distinctly, namely, one for the extraordinary pencil, which is polarized in the equatorial plane, at which the lines in the axial plane are seen distinctly; another at which the lines in the equatorial plane are seen distinctly; and, intermediate between these, a third for the ordinary pencil, which is polarized in the axial plane, at which both systems at once will be seen distinctly. And the ordinary index, which will be given by the ordinary image, will be a geometric mean between the two apparent extraordinary indices, of which one, namely, that got from the lines in the axial plane, will be the real extraordinary index.

There are two uniaxal crystals, calcite and quartz, for which we know accurately the principal refractive indices for the principal lines of the spectrum from the measures of Rudberg. The principal indices for these two minerals and the apparent indices in the two directions mentioned above are given in the following Table. The indices are given to four places of decimals, and the fixed lines C, D, E are chosen, whence the results applicable to the kinds of light most likely to be employed may be obtained, directly or by interpolation.

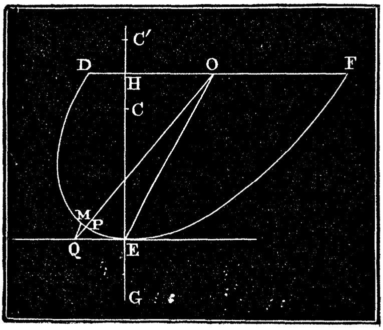
Lines.	Calcite.				Quartz.			
	μ.	μ'.	$\frac{\mu'^2}{\mu}$.	$\frac{\mu^2}{\mu'}$.	μ.	μ'.	$\frac{\mu'^2}{\mu}$.	$\frac{\mu^2}{\mu'}$.
C ...	1·6545	1·4846	1·3321	1·8438	1·5418	1·5509	1·5601	1·5328
D ...	1·6585	1·4864	1·3322	1·8505	1·5442	1·5533	1·5624	1·5352
E ...	1·6636	1·4887	1·3322	1·8590	1·5471	1·5563	1·5656	1·5380

It is well known that the double refraction of quartz differs from that of the generality of uniaxal crystals. Its wave-surface for any colour, instead of being the sphere and spheroid of Huyghens, is a surface of two distinct sheets, which, instead of touching, only make a very close approach along the axis. The polar diameters of the outer, or ordinary, and of the inner, or extraordinary, sheet differ by minute and practically equal quantities from the equatorial diameter of the ordinary sheet. The effect of this, however, on the indices, real or apparent, determined by Mr. Sorby's method on a plate cut perpendicular to the axis, would not be sensible. The peculiarity would show itself by giving the two images at different depths circularly polarized, one right-handedly and the other left-handedly.

It may be noticed that the refractive index is given by the reciprocal of the radius of curvature of a section of the wave-surface by a plane perpendicular to the lines seen in focus, and that in order that the lines may be seen distinctly, they must be perpendicular to one of the planes of principal curvature. This rule, as I proceed to show, is general; and it will much simplify the calculation in more complicated cases, by enabling us to dispense with the direct application of Huyghens's construction.

Let O be a point in the first surface of the plate, and consider a small pencil emanating from O in such a direction that its axis, after refraction, is perpendicular to the plate. With centre O describe half a wave-surface, of which only one sheet, DEF, is represented in the figure to avoid confusion. In a direction parallel to the surfaces of the plate draw a tangent plane to the wave-surface, touching it in E. Join OE, and draw EG a normal to the plate, and produce it to cut DF in H. Then OE is the course of a ray within the crystal which, after refraction, proceeds in a direction perpendicular to the plate, and is therefore the axis of the pencil. Let OPQ be an adjacent ray, cutting the wave-surface in P, and the tangent plane, which we may suppose to coincide with the second surface of the plate, in Q. Then the retardation of the wave on arriving at Q, relatively to the wave at E, will be the time the ray of light takes to travel from P to Q. The form of the wave after refraction will depend only on the value of this retardation, regarded as a function of the two coordinates which determine the position of Q on the plate. This follows at once from Huyghens's principle. If we regard QE as a small quantity of the first order, the retardation will be a small quantity of the second order; and in determining the foci of the refracted pencil we only want to know the retardation true to this order, and we may substitute for the actual retardation any quantity which bears to it a ratio that is ultimately one of equality. Hence, as the wave progresses within the crystal beyond DEF, we may feign it to be travelling in an ordinary medium, with a velocity of propagation equal to the actual wave-velocity in the direction HE normal to the plate. For if from Q we conceive a normal QM drawn to the wave-surface, the actual wave-velocity along MQ will differ from that in the direction HE by a small quantity of the first order; and since the whole distance MQ is a small quantity of the second order, we may neglect the variation of wave-velocity, and treat the medium as if it were a singly refracting one, in which a wave was travelling which had already, by some means, acquired the form DEF.

Fig. 2.



Through the normal EH draw the two rectangular planes of principal curvature of the surface at E, and let C, C′ be the centres of curvature, and ρ, ρ' the radii of curvature, on the same scale in which HE represents the wave-velocity v in the direction HE. Then the rays in that plane of principal curvature, the normals in which intersect in C, may be thought of as diverging from C in an ordinary medium, of which the refractive index is v^{-1}. If τ be the thickness of the plate, the distance of C from the second surface will be $\frac{\rho}{v}\tau$, and the product of this by v, or $\rho\tau$, will give the distance of the focus in that plane after refraction into air; and therefore the apparent index of refraction will be ρ^{-1}. Similarly ρ'^{-1} will will be the apparent index in the other plane. And in order that one or other of the two rectangular systems of lines may be seen distinctly at the proper focus, the lines must be placed perpendicular respectively to the two planes of principal curvature.

A similar construction applies to the other pencil which the plate is capable of transmitting independently, to which corresponds the other sheet of the wave-surface, and which is polarized in a plane perpendicular to the plane of polarization of the former pencil. In a biaxal crystal, in which neither sheet of the wave-surface is a sphere, there will in general be four focal distances at which lines in proper directions can be seen distinctly. For either pencil the two required directions are perpendicular to each other; and if the plate be perpendicular to one of the principal planes, or planes of optical symmetry of the crystal, the required directions of the cross lines are the same for both pencils, namely, parallel and perpendicular to the plane of symmetry.

The case next in order of simplicity to that of a uniaxal crystal cut parallel to the axis is that of a biaxal crystal cut in a direction perpendicular to one of the principal axes; but before proceeding to this it may be well to complete the investigation for a uniaxal crystal, by considering a plate cut in any manner.

Let θ be the inclination of the axis of the crystal to the normal to the plate; a, c, as before, the polar and equatorial semiaxes of the spheroid. We need only consider the extraordinary ray and the spheroid corresponding to it. Let ρ be the radius of curvature of the elliptic section made by the principal plane, ρ' the radius of curvature of the perpendicular section, which will be the length of the normal drawn as far as the axis of revolution. ρ and ρ' are to be expressed in terms of θ. We have

$$\rho^{-1}=a^{-2}c^{-2}(a^2\cos^2\theta+c\sin^2\theta)^{\frac{3}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

$$\rho'^{-1}=c^{-2}(a^2\cos^2\theta+c^2\sin^2\theta)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

(2) gives the apparent index as obtained by focusing on a line perpendicular to the principal plane, and (3) as obtained by focusing on a line in the principal plane.

We see from (2) that as θ changes from 0 to 90°, ρ changes from $a^{-1}c^2$ to $c^{-1}a^2$, of which one is greater than a and the other less than a. Hence for an intermediate value of θ $\rho=a$. For this value we have from (2)

$$\tan^2\theta=a^{\frac{2}{3}}c^{-\frac{4}{3}}(a^{\frac{2}{3}}+c^{\frac{2}{3}})=\mu^{-\frac{4}{3}}\mu'^{\frac{2}{3}}(\mu^{\frac{2}{3}}+\mu'^{\frac{2}{3}}). \quad . \quad . \quad . \quad . \qquad (4)$$

In this case, as in that of a uniaxal crystal cut perpendicular to the axis, there are only two focal distances at which a distinct image is seen. But the two cases are easily distinguished; for in the present case the ordinary and extraordinary images are both polarized in definite planes; also at one of the focal distances only one of the systems of cross lines, namely, those parallel to the principal plane, are seen distinctly; and, further, either extraordinary image becomes confused when the plate is rotated in its own plane.

For this particular inclination we have, in the case of Iceland spar, according to the indices above quoted for the line D, $\theta=53°\ 34'$.

In this mineral the normal to the plane of easy cleavage is inclined to the axis at the angle 44° 37′. Substituting this value in (2) and (3), writing μ^{-1}, μ'^{-1} for a, c, we have, for the apparent indices of the extraordinary pencil:—

		For lines perpendicular to the principal plane.	For lines parallel to the principal plane.
For the fixed line	C.....	1·5777	1·4094
	D.....	1·5809	1·4104
	E.....	1·5849	1·4116

In biaxal crystals the simplest case, and at the same time the most important, is that of a plate cut perpendicular to one of the principal axes, or so-called *axes of elasticity*. As the calculation for both pencils is precisely the same as for the extraordinary pencil in a plate of a uniaxal crystal cut parallel to the axis, it will be sufficient to give the result.

Let the principal axes be designated as those of x, y, z, to which relate the parameters or principal velocities of propagation, a, b, c, and their reciprocals, the principal indices, μ, μ', μ''. I will suppose a, b, c, taken in descending, and consequently μ, μ', μ'' in ascending order of magnitude*. For the arrangement of the Table, it will be convenient to specify the direction of the line seen in focus, and that of the normal to the plane of polarization of the image observed. When these directions are different, the plane of the plate is defined, being that containing them both.

---

* In the previous investigation for Iceland spar a was taken for the ordinary wave-velocity, in order to conform to the notation in Airy's tract, so that c was greater than a.

When they are the same, the plane of the plate may be either of the principal planes containing that common direction; and, indeed, it might be any plane containing it, only that the case of a plate cut obliquely is not at present under consideration. The apparent indices obtained by focusing will accordingly be arranged as follows:—

		Direction of line brought to focus.		
		x.	*y*.	*z*.
Direction of normal to plane of polarization.	*x*.	μ	$\frac{\mu''^2}{\mu}$	$\frac{\mu'^2}{\mu}$
	y.	$\frac{\mu''^2}{\mu'}$	μ'	$\frac{\mu^2}{\mu'}$
	z.	$\frac{\mu'^2}{\mu''}$	$\frac{\mu^2}{\mu''}$	μ''

It may be well to give the numerical results for aragonite and topaz, as calculated from Rudberg's indices. I have chosen the same fixed lines as before.

	Fixed lines.	Aragonite.			Topaz.		
		x.	*y*.	*z*.	*x*.	*y*.	*z*.
x.	C.	1·5282	1·8513	1·8420	1·6093	1·6284	1·6135
	D.	1·5301	1·8576	1·8481	1·6116	1·6296	1·6148
	E.	1·5326	1·8653	1·8554	1·6145	1·6338	1·6189
y.	C.	1·6862	1·6778	1·3919	1·6262	1·6114	1·6072
	D.	1·6902	1·6816	1·3922	1·6285	1·6137	1·6115
	E.	1·6953	1·6863	1·3929	1·6315	1·6167	1·6123
z.	C.	1·6736	1·3885	1·6820	1·6040	1·5999	1·6188
	D.	1·6773	1·3887	1·6859	1·6063	1·6041	1·6211
	E.	1·6818	1·3892	1·6908	1·6093	1·6056	1·6241

The numbers in the squares *xx*, *yy*, *zz* are, of course, the real principal indices.

I proceed now to the case of a plate cut in any direction perpendicular to one of the principal planes, to which I propose to limit myself, merely observing that the leading features of the most general case have already been noticed.

The principal plane perpendicular to the plate being a plane of optical symmetry, the proper directions for the cross lines are parallel and perpendicular to that plane. Let the plane of symmetry be the plane of xz, without necessarily implying thereby that the axis of y is that of mean parameter, and let θ be the inclination of the normal to the plate to the axis of z. The section of the wave-surface, which I assume to be that of Fresnel, by the principal plane being a circle and an ellipse, the formulæ for the foci of a line perpendicular to the principal plane will be the same as for a uniaxal crystal, inasmuch as the relation which, in the case of a uniaxal crystal, subsists between the radius of the circle and one of the semiaxes of the ellipse is not involved in the formulæ. For light polarized perpendicularly to the principal plane, then, the apparent index for a line parallel to y is given by (2), while for the other pencil it is simply b^{-1} or μ'.

To find the foci for a line lying in the plane of symmetry, we must have recourse to the wave-surface itself, and not merely to its principal section. We have to find the radius of curvature at any point in the principal section for a normal section perpendicular to the principal plane.

Let P be a point in the principal section, PN the normal at P, M a point in PN near P, and through M draw MQ parallel to y, cutting in Q the sheet of the wave-surface to which P belongs. Then the limit of $\mathrm{MQ}^2 \div 2\,\mathrm{PM}$, as M moves up to P, will be the radius of curvature required.

Taking the equation of the wave-surface under the form

$$\left.\begin{aligned}(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2\\ -b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2=0\end{aligned}\right\}, \quad . \quad . \quad . \quad (5)$$

let x, 0, z be the cordinates of P, and $x+\delta x$, y, $z+\delta z$ those of Q. Substituting in (5), which, by hypothesis, is satisfied by the coordinates x, 0, z, observing that δx, δz are small quantities of the order y^2, and omitting small quantities of the order y^4, we find

$$\left.\begin{aligned}\{a^2x^2+c^2z^2+a^2(x^2+z^2-b^2-c^2)\}2x\delta x\\ +\{a^2x^2+c^2z^2+c^2(x^2+z^2-a^2-b^2)\}2z\delta z\\ +\{a^2x^2+c^2z^2+b^2(x^2+z^2-a^2-c^2)\}y^2=0\end{aligned}\right\}$$

Let PM$=p$, and first suppose P to lie in the circular section. Then $x^2+z^2=b^2$, $x=b\sin\theta$, $z=b\cos\theta$. Also, as the normal coincides with the radius vector,

$$\frac{\delta x}{x}=\frac{\delta z}{z}=-\frac{p}{b},\ x\delta x+z\delta z=-bp.$$

Substituting in (6) and putting $y^2=2p\rho'$, we find for the curvature

$$\rho'^{-1}=\frac{b(a^2\sin^2\theta+c^2\cos^2\theta+b^2-a^2-c^2)}{b^2(a^2\sin^2\theta+c^2\cos^2\theta)-a^2c^2},$$

or

$$\rho'^{-1}=\frac{b\{(b^2-a^2)\cos^2\theta+(b^2-c^2)\sin^2\theta\}}{c^2(b^2-a^2)\cos^2\theta+a^2(b^2-c^2)\sin^2\theta}, \quad . \quad . \quad . \quad . \quad (7)$$

which gives the apparent index for light polarized in the principal plane when a line in that plane is brought into focus.

Next let P lie in the elliptic section, then

$$a^2x^2+c^2z^2=a^2c^2,$$

which reduces (6) to

$$2(x^2+z^2-b^2)(a^2x\delta x+c^2z\delta z)+\{b^2(x^2+z^2-a^2-c^2)+a^2c^2\}y^2=0, \quad .. \quad (8)$$

and

$$\frac{\delta x}{a^2x}=\frac{\delta z}{c^2z}=\frac{-p}{\sqrt{(a^4x^2+c^4z^2)}}=\frac{a^2x\delta x+c^2z\delta z}{a^4x^2+c^4z^2},$$

which, on putting ρ' for $y^2\div 2p$, reduces (8) to

$$(x^2+z^2-b^2)(a^4x^2+c^4z^2)^{\frac{1}{2}}-\{b^2(x^2+z^2-a^2-c^2)+a^2c^2\}\rho'=0. \quad . \quad (9)$$

We have also

$$\tan\theta=\frac{a^2x}{c^2z},$$

which, combined with the equation to the ellipse, gives

$$\frac{a^4x^2}{\sin^2\theta}=\frac{c^4z^2}{\cos^2\theta}=\frac{a^2c^2}{a^{-2}\sin^2\theta+c^{-2}\cos^2\theta}=a^4x^2+c^4z^2=\frac{x^2+z^2}{a^{-4}\sin^2\theta+c^{-4}\cos^2\theta}.$$

On substituting in (9), and reducing, we find

$$\rho'^{-1}=\frac{\{(b^2-a^2)\cos^2\theta+(b^2-c^2)\sin^2\theta\}(a^2\cos^2\theta+c^2\sin^2\theta)^{\frac{1}{2}}}{a^2(b^2-a^2)\cos^2\theta+c^2(b^2-c^2)\sin^2\theta}, \quad . \quad (10)$$

which gives the apparent index for light polarized perpendicularly to the principal plane, when a line in that plane is brought into focus.

To sum up. For the pencil which is polarized in the principal plane the apparent index for a line perpendicular to that plane is the real index b^{-1} or μ', while for a line in the principal plane it is given by (7). For the pencil which is polarized perpendicularly to the principal plane the apparent index for a line perpendicular to that plane is given by (2), and for a line in the plane by (10).

On examining the expressions (7) and (10) for the radii of curvature of normal sections perpendicular to the principal plane, we see that if b be the greatest or least parameter they remain constantly positive. But if b be the mean parameter, both expressions change sign twice, once in passing through zero, and once through infinity, as θ changes from 0° to 90°.

The radii of curvature become infinite together when

$$\tan^2\theta=\frac{a^2-b^2}{b^2-c^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{11}$$

that is, when the plate is perpendicular to the optic axis. For a point in the circular section the radius vanishes when

$$\tan^2\theta=\frac{c^2(a^2-b^2)}{a^2(b^2-c^2)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{12}$$

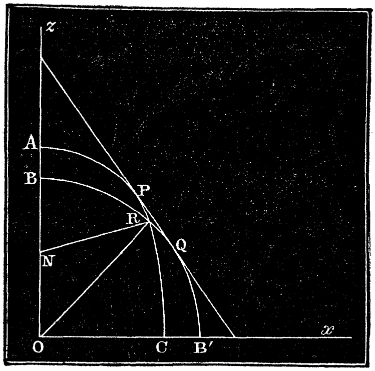
that is, when the plate is perpendicular to the ray-axis. For a point in the elliptic section the radius vanishes when

$$\tan^2\theta=\frac{a^2(a^2-b^2)}{c^2(b^2-c^2)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \tag{13}$$

that is, when the plate is perpendicular to the normal to the elliptic section at the point where the two sections intersect.

A figure may make these changes clearer. Let xOz be a quadrant of

Fig. 3.



the plane perpendicular to the axis of mean parameter. Let BB′ be the circular, and AC the elliptic section, intersecting in R, PQ the common tangent, RN a normal at R to the elliptic section. Conceive a plate cut perpendicular to the plane of xz, its normal being inclined at the angle θ to Oz; and imagine θ to change continuously from 0 to 90°: and let ρ', ρ_1' represent the radii of curvature in the secondary plane (xOz being deemed the primary plane) for points in the sections AC, BB′ respectively. As θ starts from zero, ρ' starts from a and increases, and ρ_1' starts from $b^{-1}c^2$ and decreases. When θ becomes BOR, ρ_1' vanishes, and beyond that becomes negative, while ρ' continues to increase. As θ increases to BOQ, ρ' increases positively, and ρ_1' negatively, to infinity, and beyond that both change sign, ρ' becoming negative and ρ_1' positive. As θ increases to ANR, ρ' decreases negatively to zero, while ρ_1' decreases positively from infinity. On passing ANR, ρ' becomes positive, and increases to its final value, c, which it reaches when $\theta = 90°$, while ρ_1' decreases to its final value, $b^{-1}a^2$. Thus though $a > c$ we may say that as θ increases from 0 to 90°, ρ' *increases* from a to c by passing through ∞ and 0, and ρ_1' *decreases* from $b^{-1}c^2$ to $b^{-1}a^2$ by passing through 0 and ∞.

The extravagant changes of apparent index in the immediate neighbourhood of the wave- and ray-axes could probably not well be followed by the microscope, on account of the necessity of working with pencils of finite angular aperture, which would make the phenomena of focusing blend themselves with those of conical refraction. But there can be little doubt that a large increase or diminution of apparent index on approaching the critical region would be readily discernible. That these changes are not confined to the principal plane is evident, inasmuch as one principal radius of curvature of the wave-surface becomes infinite at any point of the circle of contact of the surface with the tangent plane perpendicular to the optic axis, and one principal radius of curvature vanishes at the conical point, to whatever normal section it be thought of as belonging.

Let us now resume the equations (2), (10), which give the principal curvatures for the elliptic section, without deciding beforehand any thing as to the relative magnitude of the parameters.

As θ changes from 0 to 90°, the radius of curvature in the primary plane changes from $a^{-1}c^2$ to $c^{-1}a^2$, and that in the secondary plane from a to c; and the ratio of the radii, therefore, changes from $a^{-2}c^2$ to $c^{-2}a^2$, of which one is greater than 1 and the other less than 1. If, then, both radii remain positive, as is the case in the two principal planes passing through the mean axis, the two radii must be equal for some intermediate value of θ. Hence there must be four umbilici in each of these planes. To find the umbilici we must equate the values of ρ, ρ' given by (2), (10), whence

$$a^2c^2\{(b^2-a^2)\cos^2\theta+(b^2-c^2)\sin^2\theta\}(\cos^2\theta+\sin^2\theta)$$
$$=\{a^2(b^2-a^2)\cos^2\theta+c^2(b^2-c^2)\sin^2\theta\}(a^2\cos^2\theta+c^2\sin^2\theta),$$

which gives, after reduction,

$$\tan^4\theta=\frac{a^2(b^2-a^2)}{c^2(b^2-c^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This expression shows that the umbilici in the elliptic section made by the principal plane of greatest and least parameters are imaginary. If we take v_{xy}, v_{zy} to denote the inclinations to Oy of the normals to the umbilici in the planes of xy, zy, we have, by the requisite interchanges of letters:—

$$\tan^4 v_{xy}=\frac{a^2(a^2-c^2)}{b^2(b^2-c^2)};\quad \tan^4 v_{zy}=\frac{c^2(a^2-c^2)}{b^2(a^2-b^2)}. \quad . \quad . \quad . \quad (15)$$

If a plate be cut perpendicular to the normal at one of these umbilici, one of the polarized pencils which it transmits will give the images of both systems of cross lines distinct together; and the distinctness will not be affected by rotating the plate in its own plane while the cross lines are fixed. In this respect it agrees with a plate of a uniaxal crystal cut in an arbitrary direction, with which it might easily be confounded. But if the double refraction be strong enough to give a sensible *lateral* separation of the two oppositely polarized images, the two cases may be distinguished thereby in either of two ways:—First, if the images be compared with a mark fixed to the focus of the eyepiece, and the crystal be rotated in its own plane, while the object viewed through it is fixed, in the case of a uniaxal crystal the image free from astigmatism will remain fixed, while any point of the other describes a small circle round its mean position, whereas in a plate of a biaxal crystal cut perpendicular to the normal at one of the umbilici above considered it is the reverse; the image affected by astigmatism remains fixed, though its distinctness alters, while any point in the other describes a small circle about its mean position. Secondly, if the plane of separation of the two oppositely polarized images be noticed, in a uniaxal crystal the plane of polarization of the image which is free from astigmatism will be parallel to the plane of separation, while in a biaxal crystal cut as above supposed it will be perpendicular to the plane of separation*.

There are no umbilici in the circular sections of the wave-surface made by the principal planes. If we equate ρ' given by (7) to be the radius of curvature in the primary plane, we get, in fact, $\cos^2\theta+\sin^2\theta=0$, which cannot be satisfied.

The formulæ (15) give for v_{xy}, v_{zy} (ray D) in aragonite 69° 26′ and 45° 8′; in topaz 46° 49′ and 55° 27′.

In the employment of his method Mr. Sorby has chiefly had in view the discrimination of minerals, but it admits of one or two interesting applications to optical theory.

At the time when Fresnel invented his theory of double refraction it had been supposed, from the observations of those who had specially examined the question, that in biaxal crystals one of the rays obeyed the ordinary law of refraction; and Fresnel proved by two methods, both requiring skill on the part of the optician who cut the crystals, that the anticipation that his theory led him to entertain that that would not prove to be the case was verified. It is interesting to find that the extraordinary character of the refraction of both rays in a biaxal crystal admits of being established by such a comparatively simple mode of observation as that of Mr. Sorby.

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* [The first test supposes the surfaces of the plate to be pretty truly parallel, as otherwise it would produce displacement in consequence of its slightly wedge-shaped form; and Mr. Sorby thinks this requirement would prevent this test from being of much use. As to the second test, it is needless to observe that in any case in which we know independently which is the principal plane we need not attempt to observe the lateral separation of the images.—*October* 1877.]

The theory of Fresnel is confessedly wanting in rigour; and though the observations of Huyghens, of Wollaston, and of Malus proved that in Iceland spar Huyghens's construction, if not rigorously true, was at least a very close approximation to the truth, it seemed desirable to put it to a sharper observational test, more especially as different theories might lead to Huyghens's construction *as a near approximation.* For instance, in a paper read before the Cambridge Philosophical Society in 1849, I obtained a formula* which led me to perceive that double refraction would be simply accounted for by attributing it to a difference of inertia in different directions, such as would be produced if a fluid had to make its way among a number of bodies on the average regularly arranged, that arrangement being different in different directions, and that the wave-velocity on this theory would be related to the direction of the wave normal just as in the theory of Fresnel, with the exception that the reciprocals of wave-velocities would take the place of the velocities themselves. I refrained, however, from putting forward that theory either in the memoir referred to or elsewhere (though I have incidentally alluded to it in my report on double refraction†), because, on calculating the difference of refraction of the extraordinary ray on this theory and according to Huyghens's construction, at about 45° from the axis, where the difference would be greatest, I found it barely small enough, as seemed to me, to have escaped detection. Still this theory, which has occurred independently, in the same or a similar shape, to others‡, led me to wish for a more exact verification; and in the report referred to I have proposed a method which seemed to me well calculated to lead to the desired result. This method I carried out some years later in the case of Iceland spar, though I did not publish the results; and I found that, to the limit of error of my observations (about 0·0001 in the index), Huyghens's construction was fully confirmed, while the error of the other was nearly a hundred times as great as the limit of error of the observations§. The accuracy of the Huyghenian law has also been confirmed by the elaborate observations of M. Abria*.

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* Cambridge Philosophical Transactions, vol. viii. p. 111.

† Report of the British Association for 1862, part i. p. 269.

‡ See papers by the late Professor Rankin in the Philosophical Magazine, vol. i. (1851), p. 441, and by Lord Rayleigh in the same, vol. xli. (1871) p. 519.

§ This result is briefly mentioned in the Proceedings of the Royal Society, vol. xx. p. 443.

In the method of prismatic refraction employed by M. Abria and myself, the difference between Huyghens's construction and the result of the theory just referred to is greatest about 45° from the axis, while extremely close to the axis, or to the equator, it would hardly be sensible. Mr. Sorby's method is remarkable for this, that it brings out into prominence variations of refraction with change of direction, though the absolute refractions which are involved may be nearly the same. Thus Mr. Sorby informs me that his method shows with perfect distinctness the two widely different foci for a plate of Iceland spar cut perpendicular to the axis, even though the inclination of the rays concerned to the axis is so small that, when polarized light is used, which is extinguished by an analyzer, the field remains dark after the interposition of the crystal.

In the theory referred to above the extraordinary sheet of the wave-surface is generated by the revolution of the curve which is the envelope of straight lines whose distance, v, from the origin is connected with their inclination, $90°-\theta$, to the axis by the relation

$$v^{-2}=a^{-2}\cos^2\theta+c^{-2}\sin^2\theta.$$

The radius of curvature of this envelope at the axis is $c^{-2}(2ac^2-a^3)$, and accordingly the apparent index, m, is given by

$$m=\frac{\mu^3}{2\mu^2-\mu'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

which exceeds the apparent index, $\mu^{-1}\mu'^2$, given by the spheroid of Huyghens by

$$\frac{[(\mu^2-\mu'^2)^2}{\mu(2\mu^2-\mu'^2)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The same formula will apply to a point in the equator if we interchange μ and μ'.

Putting these excesses into numbers, according to Rudberg's indices for the line D in Iceland spar, we find 0·0536 and 0·1180, giving for apparent indices 1·3858 instead of 1·3322 for a plate perpendicular to the axis, and 1·9685 instead of 1:8505 for a plate parallel to the axis when the microscope is focused on a line in the equatorial plane. These differences are much too large to escape detection.

POSTSCRIPT.

Being anxious to complete the theory of the images free from astigmatism by determining in what cases, if in any, they could be formed by transmission in a perpendicular direction across a crystalline plate cut otherwise than perpendicular to a principal plane, I have since worked out the differential equation (between two parameters) of the lines of curvature of the wave-surface, the discussion of which shows that there are no umbilici out of the principal planes. Hence the four directions determined by equations (15) are the only ones perpendicular to which if a plate be cut one of the images is free from astigmatism.—*October* 1877.

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* 'Annales de Chimie,' tom. i, (1874) p. 289.

XI. "Notes on Physical Geology.—No. III. On the probable Age of the Continent of Asia and Europe; and on the Absolute Measure of Geological Time." By Rev. S. Haughton, M.D., D.C.L., F.R.S., Professor of Geology in the University of Dublin. Received June 11, 1877.

This paper was withdrawn by the Author after the reading, for correction of a numerical error, and will appear as amended next Session.

The Society then adjourned over the Long Vacation, to Thursday, November 15.

"On the Increase in Resistance to the Passage of an Electric Current produced on certain Wires by Stretching." By Herbert Tomlinson, B.A., Demonstrator of Natural Philosophy, King's College, London. Communicated by Prof. W. G. Adams, F.R.S. Received November 14, 1876. Read December 21 *.

The object of this iuquiry was (1) to determine the relation between increased resistance to the passage of an electric current and stretching-force; (2) to ascertain how much of the increased resistance in each case is produced by mere increase of length and diminution of section.

In order to determine the increase of resistance from stretching, the wires were each divided into two parts about 14 feet or more in length. One end of each part was fastened to a stout hook, firmly fixed into a block of wood, the two hooks used being about 8 inches apart, and the block of wood in which they were fixed securely fastened across two strong uprights, which were placed resting against the wall, so that the weights attached to the wires might swing clear of the table on which the uprights were placed. A loop was made at the other end of each part

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* See Proc. Roy. Soc. vol. xxv. p. 451.

Fig. 1.

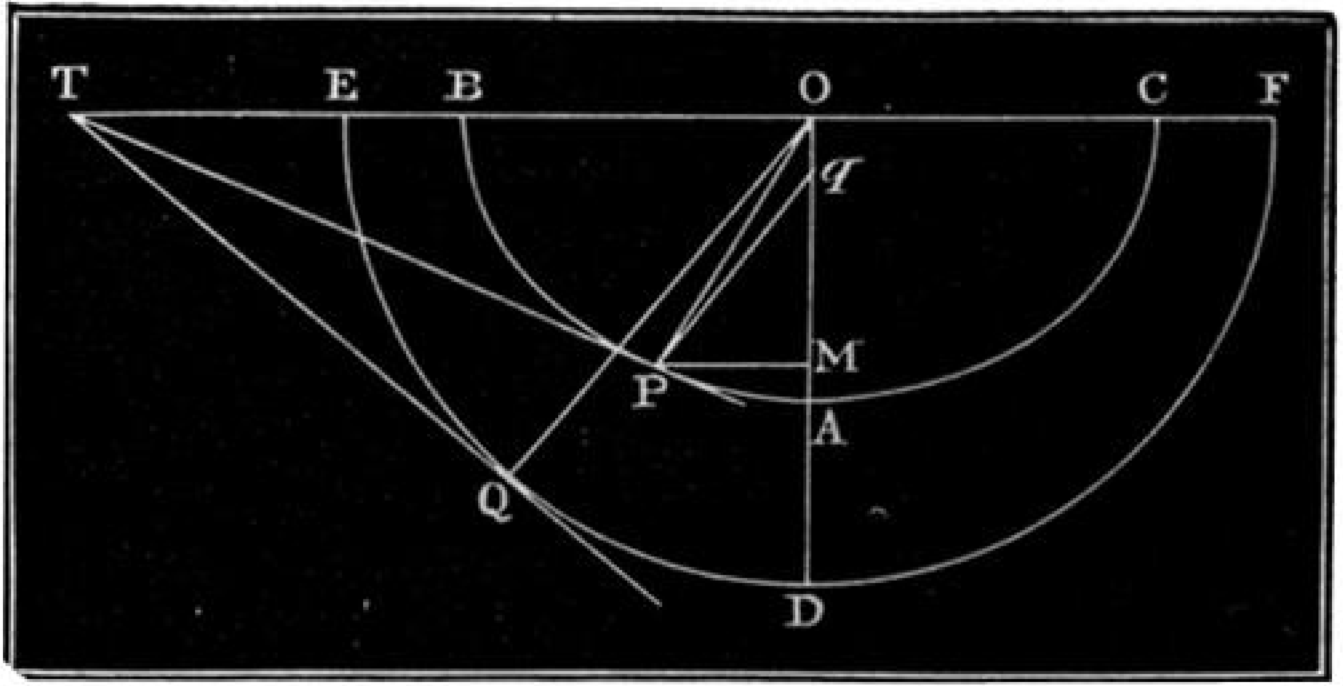

Fig. 2.

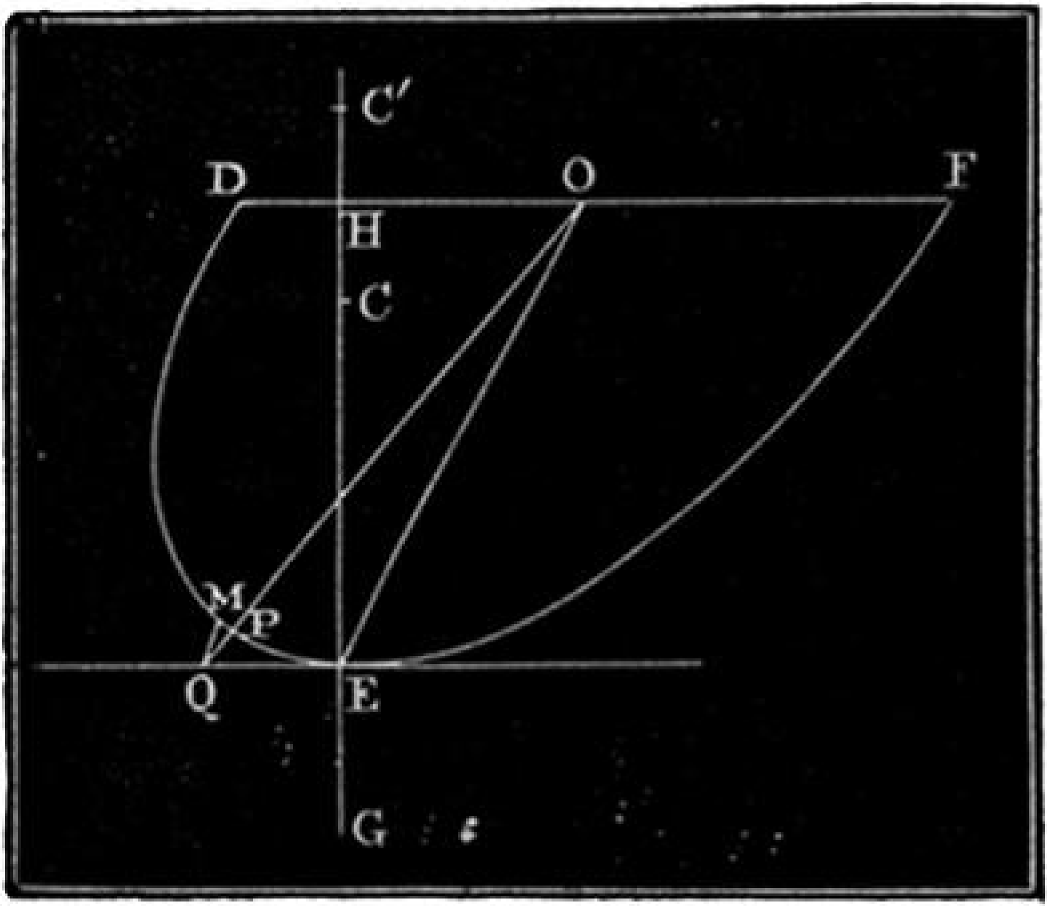

Fig. 3.